THE ENUMERATION OF EDGE COLORINGS AND HAMILTONIAN CYCLES BY MEANS OF SYMMETRIC TENSORS

PETER ZOGRAF

Abstract. Following Penrose, we introduce a family of graph functions defined in terms of contractions of certain products of symmetric tensors along the edges of a graph. Special cases of these functions enumerate edge colorings and cycles of arbitrary length in graphs (in particular, Hamiltonian cycles).

1. Introduction

The number of Hamiltonian cycles of a graph is an elusive invariant that is hard to compute. One can find several formulas in the literature that express the number of Hamiltonian cycles of a graph in terms of its adjacency matrix. However, all these formulas are rather complicated (e.g., they involve either traces of powers [6] or determinats and permanents [4] of submatrices of the adjacency matrix) and cannot be efficiently used for counting Hamiltonian cycles. The aim of this note is to propose a different approach to the enumeration of Hamiltonian cycles in graphs based on tensor contractions. The adjacency matrix enters into this construction somewhat implicitly by governing the the order in which the tensors are contracted. The same construction works for the enumeration of edge colorings and cycles of arbitrary length.

The idea of constructing graph invariants by means of tensor contractions traces back to Penrose [7], who proposed a novel (though not quite successful) approach to the 4-color problem. Later this idea found a number of remarkable applications, e.g. in the theory of Vassiliev knot invariants [5, 1] (some other applications are also mentioned in the survey [3]). However, they all deal with 3-valent embedded graphs (like 3-valent planar maps in the 4-color problem or Feynman diagrams in the theory of Vassiliev knot invariants) and make use of antisymmetric tensors (the structure tensors of Lie algebras, to be precise). Here we adapt Penrose's construction to arbitrary graphs and

2000 *Mathematics Subject Classification.* Primary 05C45; Secondary 05C15.
Key words and phrases. Graph function, edge coloring, Hamiltonian cycle.
Partially supported by the ARC grant DP0343028.

symmetric tensors and apply it to the above mentioned enumeration problems in graphs.

Acknowledgements. The author is grateful to the Institute of Mathematical Sciences at Stony Brook University for hospitality and support during 2002-03 academic year.

2. Symmetric tensors and graph functions

Let G be a finite graph (possibly with loops and multiple edges). The set of vertices of G we denote by $V(G) = \{v_1, \dots, v_n\}$, where $n = |V(G)|$ is the total number of vertices, and the set of edges of G we denote by $E(G)$. For each vertex v_i we denote by d_i its degree (or valency), $i = 1, \dots, n$. Then the number of edges of G is given by

$$|E(G)| = \frac{1}{2}\sum_{i=1}^{n} d_i.$$

Now let $\mathbb{F}$ be a field, and let $V \cong \mathbb{F}^r$ be a vector space of dimension r over $\mathbb{F}$. Fix a symmetric bilinear form $B : V \otimes V \longrightarrow \mathbb{F}$. The graph G together with the bilinear form B define a multilinear form

$$B_G : V^{\otimes d_1} \otimes \cdots \otimes V^{\otimes d_n} \longrightarrow \mathbb{F}, \tag{1}$$

which is constructed as follows. At each vertex v_i of G we place d_i-th tensor power $V^{\otimes d_i}$ of the vector space V, where the factors are labeled by the half-edges of G incident to v_i. Each edge of G defines a contraction of two copies of V (corresponding to its two half-edges) by means of the bilinear form B. We obtain the multilinear form B_G by performing such contractions over the set $E(G)$ of all edges of G. Rigorously speaking, the multilinear form B_G depends on the order of half-edges at each vertex v_i, or, equivalently, on the order of factors in the tensor power $V^{\otimes d_i}$. However, its restriction to $S^{d_1}V \otimes \cdots \otimes S^{d_n}V$, where S^dV denotes the d-th symmetric power of V, is defined uniquely.

Now fix a sequence $\mathcal{A} = \{A_1, A_2, \dots\}$ of symmetric contravariant d-valent tensors $A_d \in S^dV \subset V^{\otimes d}$. We treat the tensor product $A_{d_1} \otimes \cdots \otimes A_{d_n}$ as an element of $V^{\otimes d_1} \otimes \cdots \otimes V^{\otimes d_n}$ and consider the element

$$\mathcal{F}_{\mathcal{A},B}(G) = B_G(A_{d_1} \otimes \cdots \otimes A_{d_i}) \in \mathbb{F}. \tag{2}$$

Roughly speaking, $\mathcal{F}_{\mathcal{A},B}(G)$ is obtained by placing a copy of A_d at each vertex of G of degree d and contracting $\otimes_{i=1}^n A_{d_i}$ using B over $|E(G)|$ pairs of indices corresponding to the edges of G. Thus, to each pair $\mathcal{A}$, B, where $\mathcal{A}$ is a sequence of symmetric d-tensors ($d = 1, 2, \dots$) and B is a symmetric bilinear form, we associate an $\mathbb{F}$-valued mapping $\mathcal{F}_{\mathcal{A},B}$ on the set of isomorphism classes of graphs, or an $\mathbb{F}$-valued *graph function* in the terminology of [8].

3. Enumeration of edge colorings

Given a graph G, an *r-edge coloring* of G is a coloring of edges of G in r colors such that at each vertex $v_i \in V(G)$ all d_i edges incident to it have different colors. Clearly, an r-edge coloring exists only if $r \geq d_i$ for all $i = 1, \dots, n = |V(G)|$ and only if G contains no loops.

Let us show that the number of r-edge colorings of G can be realized as a graph function $\mathcal{F}_{\mathcal{A},B}$ for some suitably chosen $\mathcal{A}$ and B. Put $\mathbb{F} = \mathbb{R}$ and consider the usual Euclidean coordinates in $V = \mathbb{R}^r$. Take B given in these coordinates by the identity $r \times r$-matrix I_r, and define the components of the tensors A_d, $d = 1, 2, \dots,$ by the formula

$$A_d^{i_1 \dots i_d} = \begin{cases} 1 & \text{if } i_1, \dots, i_d \in \{1, \dots, r\} \text{ are pairwise distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1. *For $\mathcal{A}$ as above, the value of the graph function $\mathcal{F}_{\mathcal{A}, I_r}$ on any graph G is equal to the number of r-edge colorings of G.*

Proof. In order to compute the value $\mathcal{F}_{\mathcal{A}, I_r}(G)$ first we have to decide which products of components of A_d contribute to it non-trivially. We interprete the indices $1, \dots, r$ as colors of the half-edges of G. A product of n components $A_{d_1}^{i_1 \dots i_{d_1}} \dots A_{d_n}^{i_{m-d_n+1} \dots i_m}$ (where $n = |V(G)|$ is the number of vertices and $m = \sum_{j=1}^n d_j = 2|E(G)|$ is twice the number of edges of G) makes a non-zero contribution to $\mathcal{F}_{\mathcal{A}, I_r}(G)$ if and only if the colors agree on each edge of G or, equivalently, if and only if for each edge the both indices that label two of its half-edges are the same (otherwise the bilinear form $B = I_r$ vanishes). By definition, $A_d^{i_1 \dots i_d} \neq 0$ if and only if all the indices $i_1 \dots i_d \in \{1, \dots, r\}$ are distinct, so that the non-zero contributions are in one-to-one correspondence with r-edge colorings of G. Since every such contribution is equal to 1, the number of r-edge colorings of G is exactly $\mathcal{F}_{\mathcal{A}, I_3}(G)$. $\square$

Remark 1. The case of 3-edge colorings of 3-valent graphs (also called *Tait colorings*), is of a special interest because of its relation to the 4-color problem. Namely, a planar 3-valent map is 4-colorable if and only if it admits a Tait coloring, cf. [7, 2, 3]. Take $\mathcal{A} = \{A_3\}$ and $B = I_3$ in Theorem 1. Then for any 3-valent graph G the value $\mathcal{F}_{A_3, I_3}(G)$ is equal to the number of Tait colorings of G.

4. Enumeration of cycles

Given a graph G, by a *(multi)cycle* we understand a 2-valent subgraph C in G. Note that we do not require a cycle to be connected. We denote by $|C|$ the *length* of the cycle C (that is, the number of edges of

G that belong to C), and by $l(C)$ the number of connected componets of C. Clearly, the length of a cycle cannot exceed $n = |V(G)|$, and cycles of length n are spanning cycles in G. A *Hamiltonian cycle* is a *connected* cycle of length n.

The *type* of a cycle C in G is the partition $\lambda_C = [|C_1|, \dots, |C_l|]$ of the number $|C|$, where $C_1, \dots, C_l$ are the connected componets of C, $l = l(C)$. The weight of partition λ_C is $|\lambda_C| = |C|$, and the length is $l(\lambda_C) = l(C)$. For each partition λ we define a graph function N_λ by

$$N_\lambda(G) = \#\{C \subset G | \lambda_C = \lambda\},$$

i.e., $N_\lambda(G)$ is the number of cycles of type λ in G. In particular, $N_{[n]}(G)$ is the number of Hamiltonian cycles in G.

For k a positive integer, denote by $p_k(x_1, x_2, \dots) = x_1^k + x_2^k + \dots$ the k-th power sum in variables $x_1, x_2, \dots$. Given a partition $\lambda = [k_1, \dots, k_l]$, we define a homogeneous symmetric function p_λ of degree $|\lambda| = k_1 + \dots + k_l$ by the formula

$$p_\lambda(x_1, x_2, \dots) = \prod_{i=1}^{l} p_{k_i}(x_1, x_2, \dots).$$

We want to show that under a special choice of $\mathcal{A}$ and B the graph function $\mathcal{F}_{\mathcal{A},B}$ defined in Section 2 counts the number of cycles of any given type in graphs. We take $\mathbb{F} = \mathbb{C}$ and consider the standard coordinates in $V = \mathbb{C}^r$. In these coordinates the bilinear form B is given by the identity $r \times r$ matrix I_r. We put

$$A_1^i = \begin{cases} 0 & \text{if } i \neq r, \\ t & \text{if } i = r, \end{cases}$$

and define the tensors A_d for $d \geq 2$ componentwise by the formula

$$A_d^{i_1 \dots i_d} = \begin{cases} x_i & \text{if } (i_1 \dots i_d) \text{ is a permutation of } (i\, i\, r \dots r), \\ & \qquad\qquad\qquad\qquad\qquad i = 1, \dots, r-1, \\ t & \text{if } (i_1 \dots i_d) = (r \dots r), \\ 0 & \text{otherwise}, \end{cases}$$

where $x_1, \dots, x_{r-1}$ and t are arbitrary complex numbers. The main result of this section is the following

Theorem 2. *For $\mathcal{A} = \{A_1, A_2, \dots\}$ as above, the value of the graph function $\mathcal{F}_{\mathcal{A},I_r}$ on any graph G is given by the formula*

$$\mathcal{F}_{\mathcal{A},I_r}(G) = \sum_{|\lambda| \leq n} t^{n-|\lambda|} p_\lambda(x_1, \dots, x_{r-1})\, N_\lambda(G), \tag{3}$$

where the sum is taken over the set of all partitions λ of weight $|\lambda| \leq n = |V(G)|$.

Proof. As in the proof of Teorem 1, we interprete the indices $1, \dots, r$ as colors of the half-edges of G. Similarly, a product of n components $A_{d_1}^{i_1 \dots i_{d_1}} \dots A_{d_n}^{i_{m-d_n+1} \dots i_m}$ (where $n = |V(G)|$ and $m = \sum_{j=1}^n d_j = 2|E(G)|$) contributes non-trivially to $\mathcal{F}_{\mathcal{A}, I_r}(G)$ if and only if the colors agree on each edge of G or, equivalently, if and only if for each edge the both indices that label two of its half-edges are the same. Thus, in this case the non-zero contributions are in one-to-one correspondence with edge colorings of G in r colors with the following properties:

(i) an edge incident to a vertex of degree 1 has color r, and

(ii) at each vertex $v_j \in V(G)$ of degree $d_j \geq 2$ two edges incident to it have some color $i_j \in \{1, \dots, r\}$, and the rest $d_i - 2$ edges have color r (if an edge makes a loop we count it twice).

The closure of the union of edges with colors $1, \dots, r-1$ is a cycle C in G, and every connected component C_j of C is colored in one of the colors $i_j \in \{1, \dots, r-1\}$. The contribution to $\mathcal{F}_{\mathcal{A}, I_r}(G)$ from this coloring is $t^{n-|\lambda_C|} \prod_{j=1}^{l(C)} x_{i_j}^{|C_j|}$, where $\lambda_C = [|C_1|, \dots, |C_l|]$ is the partition associated with C and $l = l(C)$ is the number of connected components of C. Therefore, the contribution from all possible colorings of the cycle C is equal to

$$t^{n-|\lambda_C|} \prod_{j=1}^{l(C)} \left(\sum_{i=1}^{r-1} x_i^{|C_j|} \right) = t^{n-|\lambda_C|} p_{\lambda_C}(x_1, \dots, x_{r-1}),$$

and summig up the contributions from all cycles in G we get the assertion of the theorem. $\square$

Corollary 1. *The graph function $\mathcal{F}_{\mathcal{A}, I_r}$, depending on $x_1, \dots, x_{r-1}$ and t as parameters, determines the numbers $N_\lambda(G)$ uniquely for any graph G with $n \leq r-1$ vertices.*

Proof. By Theorem 2, the graph function $\mathcal{F}_{\mathcal{A}, I_r}$ with values in $\mathbb{C}$ factors through the ring $\mathbb{C}[x_1, \dots, x_{r-1}]^{S_{r-1}}$ of symmetric polynomials in $r-1$ independent variables $x_1, \dots, x_{r-1}$. It is well known that the polynomials $p_k(x_1, \dots, x_{r-1}),\ k = 1, \dots, n$, are algebraically independent in $\mathbb{C}[x_1, \dots, x_{r-1}]^{S_{r-1}}$ provided $n \leq r-1$. Therefore, in this case the graph function $\mathcal{F}_{\mathcal{A}, I_r}(G)$ determines the coefficients $N_\lambda(G)$ in (3) uniquely. $\square$

Remark 2. Since the coefficients $N_\lambda(G)$ in (3) are *non-negative* integers, we can uniquely find them out when

$$r \geq 1 + \max_{C \subset G} l(C),$$

where $l(C)$ is the number of the connected componenets of C and the maximum is taken over all cycles C in G, but we will not dwell on this here.

Below are two special cases of Theorem 2 of independent interest.

Corollary 2. *Put $r = 2$, $x_1 = 1$ and $t = 0$. Then for any graph G the value $\mathcal{F}_{\mathcal{A},\, I_2}(G)$ is the number of spanning cycles in G.*

Proof. By Theorem 2,

$$\mathcal{F}_{\mathcal{A},\, I_2}(G) = x_1^n \sum_{|\lambda|=n} N_\lambda(G).$$

□

The next statement concerns Hamiltonian cycles, or connected cycles of length $n = |V(G)|$.

Corollary 3. *Put $r = n+1$, $x_j = e^{2\pi\sqrt{-1}j/n}$, $(j = 1, \dots, n)$ and $t = 0$. Then for any graph G with n vertices the number of Hamiltonian cycles in G is equal to $\frac{1}{n}\mathcal{F}_{\mathcal{A},\, I_{n+1}}(G)$.*

Proof. In this case

$$p_k(x_1, \dots, x_n) = \begin{cases} 0, & k = 1, \dots, n-1, \\ n, & k = n, \end{cases}$$

so that by Theorem 2

$$\mathcal{F}_{\mathcal{A},\, I_{n+1}}(G) = n\, N_{[n]}(G).$$

□

Remark 3. The value $\mathcal{F}_{\mathcal{A},\, I_n+1}(G)$ can be effectively computed for any graph G as explained in Section 2. Thus, Corollary 3 provides a simple algorithm for counting the number of Hamiltonian cycles in graphs. Clearly, its computational complexity depends on the succession of tensor contractions along the edges of G. We hope to present a more detailed treatment of this problem elsewhere.

References

[1] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34**, 1995, 423-472.
[2] D. Bar-Natan, *Lie algebras and the four color theorem*, Combinatorica **17**, 1997, 43-52.
[3] S. V. Chmutov, S. V. Duzhin and A. I. Kaishev, *The algebra of 3-graphs*, Trans. Steklov Math. Inst. **221**, 1998, 157-186.
[4] I. P. Goulden and D. M. Jackson, *The enumeration of directed closed Euler trails and directed Hamiltonian circuits by Lagrangian methods*, European J. Combin. **2**, 1981, 131-212.

[5] M. Kontsevich, *Feynman diagrams and low dimensional topology*, First European Congress of Math. II, Birkhäuser, Basel, 1994, 97-121.
[6] L. M. Lihtenbaum, *On Hamiltonian cycles of a connected graph*, Cybernetics **4**, 1968, no. 4, 62-64.
[7] R. Penrose, *Applications of negative dimensional tensors*, Combinatorial mathematics and its applications (ed. D. J. A. Welsh), Academic Press, 1971, 221-244.
[8] W. T. Tutte, *Graph theory*, Addison-Wesley, 1984.

Steklov Mathematical Institute, St.Petersburg, 191011 Russia
E-mail address: zograf@pdmi.ras.ru